FAITHFULNESS OF FREE PRODUCT STATES

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Abstract. It is proved that the free product state, in the reduced free product of C*–algebras, is faithful if the states one started with are faithful.

The reduced free product of C*–algebras was introduced by Voiculescu as the appropriate construction for C*–algebras in the setting of his theory of freeness [**1**]. (See also the book [**2**]). Given a set I and for each $\iota \in I$ a unital C*–algebra A_ι with a state ϕ_ι whose GNS representation is faithful, the reduced free product construction yields the unique, unital C*–algebra A with unital embeddings $A_\iota \hookrightarrow A$ and a state ϕ on A such that

(i) $\phi|_{A_\iota} = \phi_\iota$,

(ii) $(A_\iota)_{\iota \in I}$ is free in (A, ϕ),

(iii) $A = C^*(\bigcup_{\iota \in I} A_\iota)$,

(iv) the GNS representation of ϕ is faithful on A.

We denote the reduced free product by

$$(A, \phi) = \mathop{*}_{\iota \in I} (A_\iota, \phi_\iota) \tag{1}$$

and ϕ is called the *free product state*. In this note, we prove that if ϕ_ι is faithful on A_ι for every $\iota \in I$ then ϕ is faithful on A.

Voiculescu [**1**] proved that the free product state in the analogous construction for von Neumann algebras is faithful under the hypothesis that each ϕ_ι is faithful. This implies that for the reduced free product of C*–algebras in (1), ϕ is faithful if, for each $\iota \in I$, letting $(\pi_\iota, \mathcal{H}_\iota, \xi_\iota)$

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denote the GNS construction for (A_ι, ϕ_ι), the vector ξ_ι is cyclic for the commutant, $\pi_\iota(A_\iota)'$ on $\mathcal{H}_\iota$. This is always the case if ϕ_ι is faithful and is a trace. However, there are (A_ι, ϕ_ι) with ϕ_ι faithful but ξ_ι not cyclic for $\pi_\iota(A_\iota)'$. See example 2.1.

§1. The main result.

Here is the main result, stated a little more formally.

Theorem 1.1. *Let I be a set and for each $\iota \in I$ let $A_\iota \neq \mathbf{C}$ be a unital C*–algebra and let ϕ_ι be a faithful state on A_ι. Let*

$$(A, \phi) = \mathop{*}_{\iota \in I} (A_\iota, \phi_\iota)$$

be the reduced free product of C–algebras. Then the state ϕ is faithful on A.*

The hypothesis that $A_\iota \neq \mathbf{C}$ is not really a restriction, because for any C*–algebra B with state ϕ, $(B, \phi) * (\mathbf{C}, \mathrm{id}) = (B, \phi)$.

The proof of the theorem will rely on Voiculescu's construction of the reduced free product, which we now briefly review. As above, let $(\pi_\iota, \mathcal{H}_\iota, \xi_\iota) = \mathrm{GNS}(A_\iota, \phi_\iota)$ and let

$$\overset{o}{\mathcal{H}}_\iota = \mathcal{H}_\iota \ominus \mathbf{C}\xi_\iota.$$

For $\zeta \in \mathcal{H}_\iota$ and $a \in A_\iota$ we will always write $a\zeta$ instead of $\pi_\iota(a)\zeta$. Set

$$\mathcal{H} = \mathbf{C}\xi \oplus \bigoplus_{\substack{n \in \mathbf{N} \\ \iota_1, \ldots, \iota_n \in I \\ \iota_j \neq \iota_{j+1}}} \overset{o}{\mathcal{H}}_{\iota_1} \otimes \overset{o}{\mathcal{H}}_{\iota_2} \otimes \cdots \otimes \overset{o}{\mathcal{H}}_{\iota_n}. \tag{2}$$

In Voiculescu's construction, each A_ι is represented on $\mathcal{H}$ and A is the C*–algebra generated by the union these A_ι acting on $\mathcal{H}$. Then ϕ is the vector state for ξ.

Let $n \in \mathbf{N}$ and $\iota_1, \ldots, \iota_n \in I$ be such that $\iota_1 \neq \iota_2$, $\iota_2 \neq \iota_3$, $\ldots$, $\iota_{n-1} \neq \iota_n$. For each $1 \leq j \leq n-1$ let $\zeta_j \in \overset{o}{\mathcal{H}}_{\iota_j}$. Define the isometry

$$V = V_{(\zeta_1, \ldots, \zeta_{n-1}, \iota_n)} : \mathcal{H}_{\iota_n} \to \mathcal{H}$$

by

$$V\xi_{\iota_n} = \zeta_1 \otimes \zeta_2 \otimes \cdots \otimes \zeta_{n-1}$$
$$V\zeta = \zeta_1 \otimes \zeta_2 \otimes \cdots \otimes \zeta_{n-1} \otimes \zeta \quad (\zeta \in \overset{o}{\mathcal{H}}_{\iota_n}).$$

Whenever B is a C*–algebra and ψ a state on B we denote by $\hat{b}$ the corresponding element in $L^2(B, \psi)$. Also, we employ the standard notation $A_\iota^{\mathrm{o}} = \ker \phi_\iota$.

Lemma 1.2. *Let $V = V_{(\zeta_1,\dots,\zeta_{n-1},\iota_n)}$ be as above. Let $m \in \mathbf{N}$, $k_1,\dots,k_m \in I$ be such that $k_1 \neq k_2, \quad k_2 \neq k_3, \quad \dots, \quad k_{m-1} \neq k_m$ and let $a_j \in A^o_{k_j}$, $(1 \leq j \leq m)$.*

If $m = 2p-1$ for an integer p with $1 \leq p < n$ and if

$$k_m = \iota_1 = k_1,\ k_{m-1} = \iota_2 = k_2,\ \dots,\ k_{p+1} = \iota_{p-1} = k_{p-1},\ k_p = \iota_p$$

then $V^* a_1 a_2 \dots a_m V = c1$ where c is the scalar

$$\begin{aligned} c =& \big(\langle a_m\zeta_1, \xi_{\iota_1}\rangle \langle a_{m-1}\zeta_2, \xi_{\iota_2}\rangle \cdots \langle a_{p+1}\zeta_{p-1}, \xi_{\iota_{p-1}}\rangle\big)\langle a_p\zeta_p, \zeta_p\rangle \cdot \\ &\cdot \big(\langle a_{p-1}\xi_{\iota_{p-1}}, \zeta_{p-1}\rangle \langle a_{p-2}\xi_{\iota_{p-2}}, \zeta_{p-2}\rangle \cdots \langle a_1\xi_{\iota_1}, \zeta_1\rangle\big). \end{aligned}$$

If $m = 2n-1$ and if

$$k_m = \iota_1 = k_1,\ k_{m-1} = \iota_2 = k_2,\ \dots,\ k_{n+1} = \iota_{n-1} = k_{n-1},\ k_n = \iota_n$$

then $V^* a_1 a_2 \dots a_m V = c a_n$ where c is the scalar

$$\begin{aligned} c =& \big(\langle a_m\zeta_1, \xi_{\iota_1}\rangle \langle a_{m-1}\zeta_2, \xi_{\iota_2}\rangle \cdots \langle a_{n+1}\zeta_{n-1}, \xi_{\iota_{n-1}}\rangle\big) \cdot \\ &\cdot \big(\langle a_{n-1}\xi_{\iota_{n-1}}, \zeta_{n-1}\rangle \langle a_{n-2}\xi_{\iota_{n-2}}, \zeta_{n-2}\rangle \cdots \langle a_1\xi_{\iota_1}, \zeta_1\rangle\big). \end{aligned} \tag{3}$$

Otherwise, $V^* a_1 a_2 \dots a_m V = 0$.

Proof. We use induction on n. If $n = 1$ then $V(\mathcal{H}_{\iota_1})$ is the canonical copy $\mathcal{H}_{\iota_1} \cong \mathbf{C}\xi \oplus \overset{o}{\mathcal{H}}_{\iota_1}$ of $\mathcal{H}_{\iota_1}$ in $\mathcal{H}$. If $m = 1$ and $k_m = \iota_1$ then $V^* a_1 V = a_1$, whereas if $m > 1$ or if $m = 1$ but $k_m \neq 1$ then one easily sees that

$$\begin{aligned} a_1 \cdots a_m \xi =& \hat{a}_1 \otimes \cdots \otimes \hat{a}_m \quad \perp (\mathbf{C}\xi \oplus \overset{o}{\mathcal{H}}_{\iota_1}) \\ a_1 \cdots a_m \overset{o}{\mathcal{H}}_{\iota_1} =& \hat{a}_1 \otimes \cdots \otimes \hat{a}_m \otimes \overset{o}{\mathcal{H}}_{\iota_1} \quad \perp (\mathbf{C}\xi \oplus \overset{o}{\mathcal{H}}_{\iota_1}), \end{aligned}$$

so that $V^* a_1 \cdots a_m V = 0$.

Now suppose $n > 1$. If $k_m \neq \iota_1$ then

$$\begin{aligned} a_1 \cdots a_m(\zeta_1 \otimes \cdots \otimes \zeta_{n-1}) =& \hat{a}_1 \otimes \cdots \hat{a}_m \otimes \zeta_1 \otimes \cdots \otimes \zeta_{n-1} \quad \perp V\mathcal{H}_{\iota_n} \\ a_1 \cdots a_m(\zeta_1 \otimes \cdots \otimes \zeta_{n-1} \otimes \overset{o}{\mathcal{H}}_{\iota_n}) =& \hat{a}_1 \otimes \cdots \hat{a}_m \otimes \zeta_1 \otimes \cdots \otimes \zeta_{n-1} \otimes \overset{o}{\mathcal{H}}_{\iota_n} \quad \perp V\mathcal{H}_{\iota_n}. \end{aligned}$$

So $k_m \neq \iota_1$ implies $V^* a_1 \cdots a_m V = 0$. Taking adjoints we see that also $k_1 \neq \iota_1$ implies $V^* a_1 \cdots a_m V = 0$. Hence suppose $k_m = \iota_1 = k_1$. Then $m = 1$ or $m \geq 3$. Furthermore,

$$\begin{aligned} a_1 \cdots a_m(\zeta_1 \otimes \cdots \otimes \zeta_{n-1}) =& \hat{a}_1 \otimes \cdots \hat{a}_{m-1} \otimes \big(a_m\zeta_1 - \langle a_m\zeta_1, \xi_{\iota_1}\rangle\xi_{\iota_1}\big) \otimes \zeta_2 \cdots \otimes \zeta_{n-1} + \\ &+ \langle a_m\zeta_1, \xi_{\iota_1}\rangle a_1 \cdots a_{m-1}(\zeta_2 \otimes \cdots \otimes \zeta_{n-1}) \end{aligned} \tag{4}$$

and similarly for every element of $\zeta_1 \otimes \cdots \otimes \cdots \zeta_{n-1} \otimes \overset{o}{\mathcal{H}}_{\iota_n}$. If $m = 1$ then the second term of the right hand side of (4) is perpendicular to $V\mathcal{H}_{\iota_n}$ and taking V^* of (4) shows

$$V^* a_1 V = \langle a_1 \zeta_1, \zeta_1 \rangle 1.$$

If $m \geq 3$ then the first term of the right hand side of (4) is perpendicular to $V\mathcal{H}_{\iota_n}$ and we see that

$$V^* a_1 \cdots a_m V = \langle a_m \zeta_1, \xi_{\iota_1} \rangle V^* a_1 \cdots a_{m-1} U,$$

where $U = V_{(\zeta_2,\ldots,\zeta_{n-1},\iota_n)}$. In a like manner we show that

$$\begin{aligned} V^* a_1 \cdots a_{m-1} U &= (U^* a_{m-1}^* \cdots a_1^* V)^* = \overline{\langle a_1^* \zeta_1, \xi_{\iota_1} \rangle} (U^* a_{m-1}^* \cdots a_2^* U)^* \\ &= \langle a_1 \xi_{\iota_1}, \zeta_1 \rangle U^* a_2 \cdots a_{m-1} U. \end{aligned}$$

Now applying the inductive hypothesis finishes the proof.

□

Lemma 1.3. *Let (A, ϕ) be as in Theorem 1.1. Let $n \in \mathbf{N}$ and $\iota_1, \ldots, \iota_n \in I$ be such that $\iota_1 \neq \iota_2$, $\iota_2 \neq \iota_3$, $\ldots$, $\iota_{n-1} \neq \iota_n$. For each $1 \leq j \leq n-1$ let $\zeta_j \in \overset{o}{\mathcal{H}}_{\iota_j}$. Let $V = V_{(\zeta_1,\ldots,\zeta_{n-1},\iota_n)}$ be as defined before Lemma 1.2. Then $V^* A V = A_{\iota_n}$.*

Proof. Clearly $V^* V = 1 \in A_{\iota_n}$ and from Lemma 1.2, $V^* a_1 \cdots a_m V \in A_{\iota_n}$ for each $a_1 \cdots a_m$ as in the statement of Lemma 1.2. Since the collection of all these $a_1 \cdots a_m$ together with 1 spans a dense subspace of A, it follows that $V^* A V \subseteq A_{\iota_n}$. In order to see the other inclusion, using (3) it is enough to see that for every $1 \leq j \leq n-1$ one can find $a_j, a_{2n-j} \in A_{\iota_j}^{\mathrm{o}}$ such that

$$\langle a_j \xi_{\iota_j}, \zeta_j \rangle \neq 0, \qquad \langle a_{2n-j} \zeta_j, \xi_{\iota_j} \rangle \neq 0.$$

But this can be done because ξ_{ι_j} is cyclic for the action of A_{ι_j} on $\mathcal{H}_{\iota_j}$.

□

Proof of Theorem 1.1. Suppose for contradiction there is $a \in A$ with $a \geq 0$, $a \neq 0$ and $\phi(a) = 0$. Then

$$\langle a\xi, \xi \rangle = \phi(a) = 0.$$

If $n \in \mathbf{N}$ and $\iota_1, \ldots, \iota_n \in I$ with $\iota_1 \neq \iota_2$, $\iota_2 \neq \iota_3$, $\ldots$, $\iota_{n-1} \neq \iota_n$, then let $p_{\iota_1,\iota_2,\ldots,\iota_n}$ denote the projection from $\mathcal{H}$ onto the direct summand

$$\overset{o}{\mathcal{H}}_{\iota_1} \otimes \overset{o}{\mathcal{H}}_{\iota_2} \otimes \cdots \otimes \overset{o}{\mathcal{H}}_{\iota_n}$$

in (2). Since $a \geq 0$ and $a \neq 0$, for some $n \in \mathbf{N}$ one can choose $\iota_1, \ldots, \iota_n$ as above such that

$$p_{\iota_1,\ldots,\iota_n} a p_{\iota_1,\ldots,\iota_n} \neq 0.$$

Let n be least for which such a choice can be made. Then there are $\zeta_j \in \overset{o}{\mathcal{H}}_{\iota_j}$, $(1 \leq j \leq n-1)$, such that letting $V = V_{(\zeta_1,\ldots,\zeta_{n-1},\iota_n)}$ we have $V^*aV \neq 0$. However, $V^*aV \geq 0$ and by Lemma 1.3 $V^*aV \in A_{\iota_n}$. Since ϕ_{ι_n} is faithful on A_{ι_n}, we must have

$$0 < \phi_{\iota_n}(V^*aV) = \langle V^*aV\xi_{\iota_n}, \xi_{\iota_n}\rangle. \tag{5}$$

If $n = 1$ then

$$\langle V^*aV\xi_{\iota_n}, \xi_{\iota_n}\rangle = \langle a\xi, \xi\rangle = \phi(a) = 0,$$

contradicting (5). If $n > 1$ then from (5) and the definition of V we have

$$0 < \langle V^*aV\xi_{\iota_n}, \xi_{\iota_n}\rangle = \langle a(\zeta_1 \otimes \cdots \otimes \zeta_{n-1}), \zeta_1 \otimes \cdots \otimes \zeta_{n-1}\rangle.$$

Hence $p_{\iota_1,\ldots,\iota_{n-1}} a p_{\iota_1,\ldots,\iota_{n-1}} \neq 0$, which contradicts the choice of n.

□

§2. An example.

In this section we present an example of a C*–algebra A with a faithful state ϕ such that in the GNS representation the standard vector is not cyclic for the action of the commutant.

Example 2.1. Let $\mathcal{T}$ be the Toeplitz algebra generated by the nonunitary isometry, S, and let $\pi : \mathcal{T} \to C(\mathbf{T})$ be a unital $*$–homomorphism. Let $\mathrm{ev} : C(\mathbf{T}) \to \mathbf{C}$ be a character of $C(\mathbf{T})$ and let $\psi_0 = \mathrm{ev} \circ \pi$. Let ψ_1 be any faithful state on $\mathcal{T}$. Let tr_2 be the tracial state on $M_2(\mathbf{C})$ and let ρ be a pure state on $M_2(\mathbf{C})$. Let $A = \mathcal{T} \otimes M_2(\mathbf{C})$ and let

$$\phi = \tfrac{1}{2}\psi_1 \otimes \mathrm{tr}_2 + \tfrac{1}{2}\psi_0 \otimes \rho : A \to \mathbf{C}.$$

Clearly, ϕ is a faithful state on A. However, letting

$$(\pi, \mathcal{H}, \xi) = \mathrm{GNS}(A, \phi),$$

we will show that $\pi(A)'\xi \neq \mathcal{H}$.

Our first task is to find $\mathcal{H}$. Let $(e_{ij})_{i,j\in\mathbf{N}}$ be a system of matrix units for the copy, $\mathcal{K}$, of the compact operators which is an ideal of $\mathcal{T}$, and let $(f_{ij})_{1\le i,j\le 2}$ be a system of matrix units for $M_2(\mathbf{C})$ such that $\rho(f_{11})=1$. Then

$$L^2(A,\psi_1\otimes\mathrm{tr}_2)\text{ has orthonormal basis }\{\psi_1(e_{jj})^{-1/2}\hat{e}_{ij}\mid i,j\in\mathbf{N}\}\otimes\{\sqrt{2}\hat{f}_{ij}\mid 1\le i,j\le 2\}$$
$$L^2(A,\psi_0\otimes\rho)\text{ has orthonormal basis }\hat{1}\otimes\{\hat{f}_{11},\hat{f}_{21}\}.$$

Let

$$\mathcal{H}'=L^2(A,\psi_1\otimes\mathrm{tr}_2)\oplus L^2(A,\psi_0\otimes\rho)$$

with norm $||\zeta_1\oplus\zeta_0||^2=\frac{1}{2}(||\zeta_1||^2+||\zeta_0||^2)$. Since for every $a\in A$ the elements $\hat{a}\in L^2(A,\phi)$ and $\hat{a}\oplus\hat{a}\in\mathcal{H}'$ have the same norm, there is an isometry $V:L^2(A,\phi)\to\mathcal{H}'$ such that $V\hat{a}=\hat{a}\oplus\hat{a}$. Let us show that V is onto $\mathcal{H}'$. Clearly

$$V(e_{ij}\otimes f_{kl})\hat{}=(\hat{e}_{ij}\otimes\hat{f}_{kl})\oplus 0.$$

Thus $L^2(A,\psi_1\otimes\mathrm{tr}_2)\oplus 0$ is in the range of V. But for $i=1,2$

$$V(\hat{1}\otimes\hat{f}_{i1})\hat{}=(\hat{1}\otimes\hat{f}_{i1})\oplus(\hat{1}\otimes\hat{f}_{i1}),$$

so $0\oplus(\hat{1}\otimes\hat{f}_{i1})$ is in the range of V. Hence V is onto $\mathcal{H}'$. Consequently, we may identify $\mathcal{H}$ with $\mathcal{H}'$ and $\xi\in\mathcal{H}$ with

$$(\hat{1}\otimes\hat{1})\oplus(\hat{1}\otimes\hat{e}_{ii})\in\mathcal{H}'.$$

We see that $L^2(\mathcal{T},\psi_1)$ can be identified with $l^2(\mathbf{N})\otimes l^2(\mathbf{N})$ by

$$L^2(\mathcal{T},\psi_1)\ni\psi_1(e_{jj})^{-1/2}\hat{e}_{ij}\mapsto\delta_i\otimes\delta_j,$$

and the left action of $\mathcal{T}$ on $L^2(\mathcal{T},\psi_1)$ is realized on $l^2(\mathbf{N})\otimes l^2(\mathbf{N})$ by $S(\delta_i\otimes\delta_j)=\delta_{i+1}\otimes\delta_j$. Moreover, identifying $L^2(M_2(\mathbf{C}),\mathrm{tr}_2)$ with $l^2(\{1,2\})\otimes l^2(\{1,2\})$ by $\sqrt{2}\hat{f}_{ij}\mapsto\delta_i\otimes\delta_j$, and identifying $L^2(A,\psi_0\otimes\rho)$ with $l^2(\{1,2\})$ in the obvious way, we identify $\mathcal{H}$ with

$$\big(l^2(\mathbf{N})\otimes l^2(\{1,2\})\otimes l^2(\{1,2\})\otimes l^2(\mathbf{N})\big)\oplus l^2(\{1,2\}).$$

Now we find $\pi(A)'$ by first finding $\pi(A)''$. Clearly

$$\pi(A)''\supseteq\pi(\mathcal{K}\otimes M_2(\mathbf{C}))''=\big(B(l^2(\mathbf{N}))\otimes B(l^2(\{1,2\}))\otimes 1\otimes 1\big)\oplus 0$$

Moreover, considering now $\pi(1\otimes M_2(\mathbf{C}))$ we see that

$$\pi(A)''=\big(B(l^2(\mathbf{N}))\otimes B(l^2(\{1,2\}))\otimes 1\otimes 1\big)\oplus B(l^2(\{1,2\})).$$

Therefore

$$\pi(A)'=\big(1\otimes 1\otimes B(l^2(\{1,2\}))\otimes B(l^2(\mathbf{N}))\big)\oplus\mathbf{C}$$

and thus $0\oplus(\hat{1}\otimes\hat{f}_{21})\perp\pi(A)'\xi$, which was what we wanted to show.

References

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